6. Curve

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GSAEK

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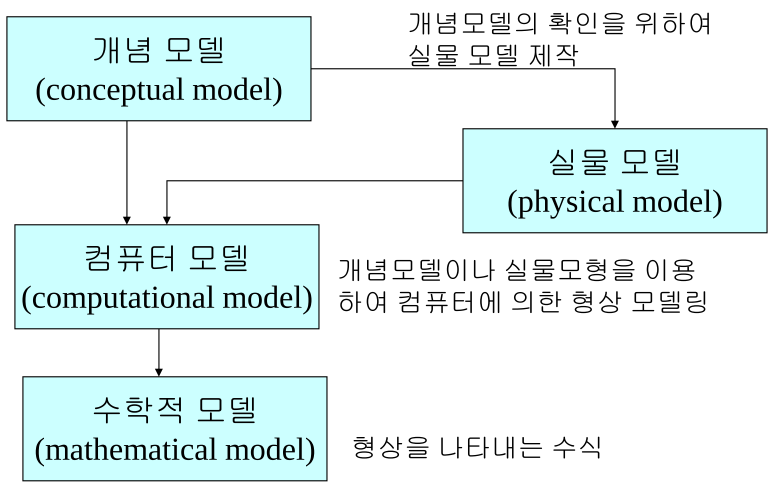
Geometric Model



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Computational Model의 종류

- **Point Model :** 물체를 점으로 근사화
- **Curve Model :** 물체를 곡선으로 표현
- **Surface Model :** 물체를 면으로 정의
- **Solid Model :** 물체의 입체형상을 정의
- **Hybrid Model :** 하나의 모델로 점, 곡선, 곡면, 입체를 모두 이용

0-3

수학적 표현식의 분류

- 비매개 변수식**(non-parametric equation)**
  - 음 함수 **(implicit form) : f(x,y) = 0**
  - 양 함수 **(explicit form) : y = f(x)**

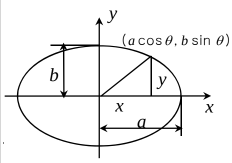
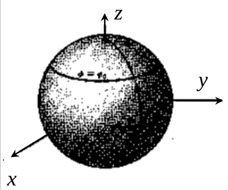
- 매개 변수식 **(parametric equation):**
  - **P(u) = [x(u), y(u), z(u)]**

0-4

수학적 모델의 표현 방법

대상	양 함수식 표현 (explicit form)	음 함수식 표현 (implicit form)	매개 변수식 표현 (parametric form)
이차원 평면 곡선	$y = f(x)$	$f(x, y) = 0$	$\mathrm{r}(t) = (x(t), y(t))$
삼차원 공간 곡선	$\begin{cases} z = f(x,y) \\ z = g(x,y) \end{cases}$	$\begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases}$	$\mathrm{r}(t) = (x(t), y(t), z(t))$
곡면	$z = f(x,y)$	$f(x,y,z) = 0$	$\mathrm{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

Curve, Surface의 표현 방법

	양함수식 표현	음함수식 표현	매개변수식 표현
	$y = b\sqrt{1 - x^2/a^2}$ $y = -b\sqrt{1 - x^2/a^2}$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$	$\mathrm{r}(\theta) = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}$ $0 < \theta < 2\pi$
	$z = \sqrt{r^2 - x^2 - y^2}$ $z = -\sqrt{r^2 - x^2 - y^2}$	$x^2 + y^2 + z^2 - r^2 = 0$	$\mathrm{r}(\theta,\phi) = \begin{pmatrix} r\cos\phi\cos\theta \\ r\cos\phi\sin\theta \\ r\sin\phi \end{pmatrix}$

Conic Section Curves

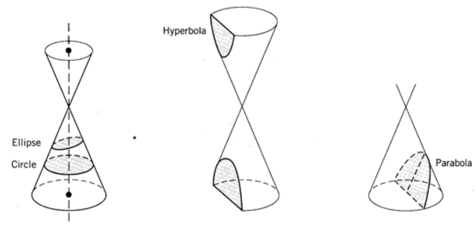
Line(직선): $ax + by - c = 0$

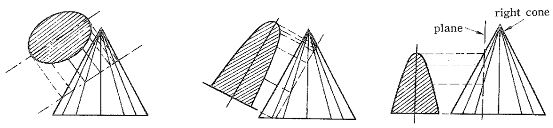
Circle(원): $x^2 + y^2 - r^2 = 0$

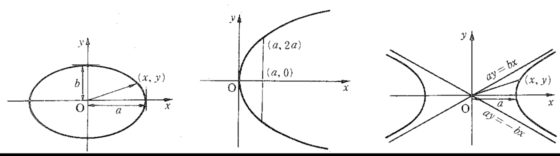
Ellipse(타원): $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

Parabola(포물선): $y^2 - 4ax = 0$

Hyperbola(쌍곡선): $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$







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비 매개변수 식의 장단점

- 장점:
  - 특별한 경우 직관적 해석이 편리하다
- 단점:
  - 하나의 형상 식이 좌표계에 의하여 변화되거나 표현 할 수 없는 경우가 생긴다.
  - 좌표계가 달라지면 형상 표현에 현실적인 어려움이 있다.
  - 곡선이나 곡면이 평면에 있지 않거나 경계가 주어진 경우에는 그 표현이 어렵거나 불가능 하다.

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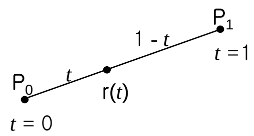
Parametric Curve

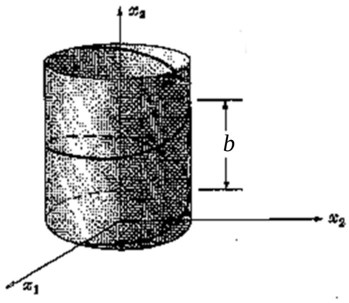
직선: $\begin{cases} x(t) = (1-t)x_0 + tx_1 \\ y(t) = (1-t)y_0 + ty_1 \end{cases}$

$$\Rightarrow \mathrm{r}(t) = (1-t)\mathrm{P}_0 + t\mathrm{P}_1$$



나선(helix)

$$\mathrm{r}(\theta) = (a \cos\theta, a \sin\theta, \frac{b}{2\pi}\theta)$$



매개 변수식의 장점

- 순차적으로 표현하기 쉽다
  - **Computer Graphics, NC Tool Path Generation**에 편리
- **2D/3D** 곡선, 곡면의 표현 형태가 비슷하다
- 자유곡선/ 곡면의 표현이 용이하다
- 이동, 회전,**Scaling**과 같은 변환이 쉽다
- 범위가 지정된 형상을 표현하기 쉽다.
- 형상을 벡터와 행렬에 의하여 쉽게 표현할 수 있다.

→ Computer를 이용한 처리 용이

Curve Model: Parametric Polynomial

- 초월함수(sin, cos, log, ...)를 쓰지 않고 다항식을 씀

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

- 자유곡선의 표현에 적합, CAD/CAM 시스템에서 가장 널리 이용

- Parametric polynomial curve의 종류
  - Power basis Polynomial Curve
  - Hermite Curve
  - Bezier Curve
  - B-Spline Curve

  * 이들은 모두 상호 변환 가능함

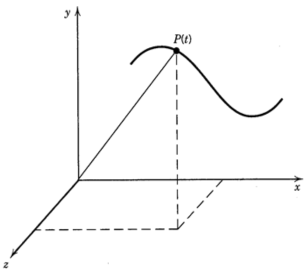
Parametric Cubic Curve (3차)

- Scalar form

$$x(u) = a_{0x} + a_{1x}u + a_{2x}u^2 + a_{3x}u^3$$
$$y(u) = a_{0y} + a_{1y}u + a_{2y}u^2 + a_{2y}u^3 \qquad 0 \le u \le 1$$
$$z(u) = a_{0z} + a_{1z}u + a_{2z}u^2 + a_{3z}u^3$$

- Vector form

$$\mathrm{r}(u) = \mathrm{a}_0 + \mathrm{a}_1 u + \mathrm{a}_2 u^2 + \mathrm{a}_3 u^3 = \sum_{i=0}^{3} u^i \mathrm{a}_i$$



- Matrix form

$$\mathrm{r}(t) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} \mathrm{a}_0 \\ \mathrm{a}_1 \\ \mathrm{a}_2 \\ \mathrm{a}_3 \end{bmatrix} = \mathrm{UA}$$

- 장점: 계산 속도가 빠르다.

  단점: 계수 $[\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3]$의 의미를 파악하기 어렵다.

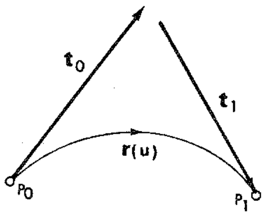
Hermite Curve

- Polynomial 곡선식

$$r(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 \qquad 0 \le u \le 1$$

- 곡선 양단에서의 끝점(P_0, P_1) 과 접선벡타(t_1, t_2)를 代入

$$\begin{aligned} r(0) &= a_0 &&= P_0 \\ r(1) &= a_0 + a_1 + a_2 + a_3 &&= P_1 \\ r'(0) &= a_1 &&= t_0 \\ r'(1) &= a_1 + 2a_2 + 3a_3 &&= t_1 \end{aligned}$$



- a_i 에 대하여 풀면 (algebraic coefficients)

$$\begin{aligned} \mathbf{a}_0 &= \mathbf{r}(0) \\ \mathbf{a}_1 &= \mathbf{r}'(0) \\ \mathbf{a}_2 &= -3\mathbf{r}(0) + 3\mathbf{r}(1) - 2\mathbf{r}'(0) - \mathbf{r}'(1) \\ \mathbf{a}_3 &= 2\mathbf{r}(0) - 2\mathbf{r}(1) + 2\mathbf{r}'(0) + \mathbf{r}'(1) \end{aligned}$$

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Hermite Curve

- 위의 식들을 대입하여 r(u)로 표시

$$\begin{aligned} \Rightarrow r(u) &= P_0 + t_0 u + [3(P_1 - P_0) - 2t_0 - t_1]u^2 + [2(P_0 - P_1) + t_0 + t_1]u^3 \\ &= (1 - 3u^2 + 2u^3)P_0 + (3u^2 - 2u^3)P_1 + (u - 2u^2 + u^3)t_0 + (-u^2 + u^3)t_1 \end{aligned}$$

Geometric coefficients : P(0), P(1), P'(0), P'(1)
Blending functions

- **Matrix form**

$$\Rightarrow r(u) = [1 \quad u \quad u^2 \quad u^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ t_0 \\ t_1 \end{bmatrix}$$

$$= UCS$$

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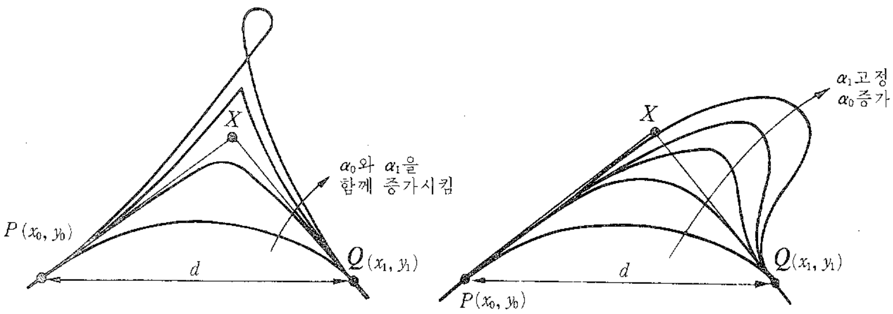
접선벡터의 영향



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Bezier Curve

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Bezier Curves

❑ History of Bezier Curve

- Bezier designed the Bezier curve
  - in the early 1960s
  - at Renault, French automobile company
  - UNISURF: surface modeler used by Renault since 1972 to design auto-bodies
- de Casteljau at Citroen also designed the Bezier curve at the same time with Bezier's

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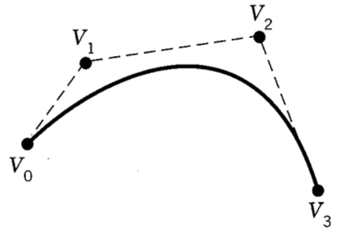
Bezier Curves

❑ Features of Bezier Curve

- The Bezier curve passes through the first and last control points
- The Bezier curve is tangent to the lines joining the first two and last two control points
- No oscillation



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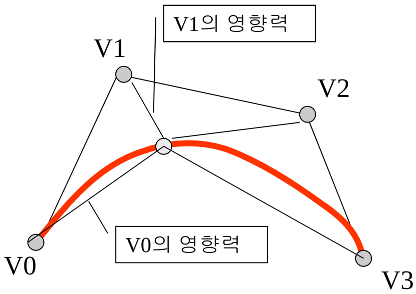
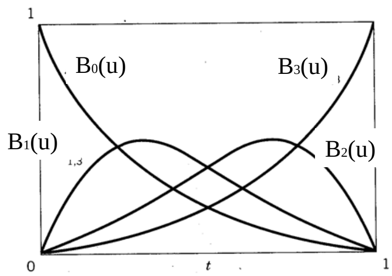
Bezier Curve의 정의

- 4개의 조정 점으로부터 영향력의 정도를 나타내는 블렌딩 함수 **(Bernstein Blending function)**를 이용하여 하나의 **Bezier Curve**를 정의





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Cubic Bezier Curve

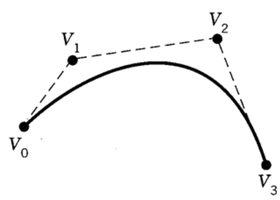
- 점 4개로 하나의 **Bezier Curve**를 정의

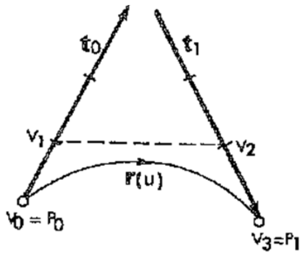
$$r(u) = (1-u)^3 V_0 + 3u(1-u)^2 V_1 + 3u^2(1-u) V_2 + u^3 V_3$$
$$= UMR \qquad 0 \le u \le 1$$

$$U = [\,1 \quad u \quad u^2 \quad u^3\,]$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$; control points





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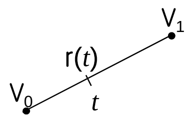
Bernstein Basis 에 의한 Bezier

- 선형 Bezier 선

$$\mathbf{r}(t) = (1-t)\mathbf{V}_0 + t\mathbf{V}_1$$



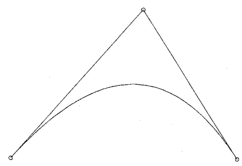
- 2차 Bezier 곡선

$$r(u) = (1-u)^2 V_0 + 2(1-u)uV_1 + u^2V_2$$

- n차 Bezier curve의 식

$$r(u) = \sum_{i=0}^{n} B_i^n(u) V_i, \qquad 0 \le u \le 1$$

$$B_i^n(u) = \binom{n}{i}(1-u)^{n-i}u^i$$



; Bernstein basis function

Bezier Curve의 일반식 (1)

- n차 Bezier curve의 식

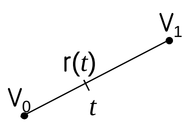
$$r(u) = \sum_{i=0}^{n} B_i^n(u) V_i, \qquad 0 \le u \le 1$$

여기서, $B_i^n(u) = \binom{n}{i}(1-u)^{n-i}u^i$; Bernstein basis function

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \qquad i = 0, \cdots, n$$

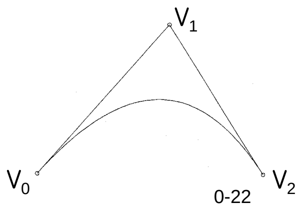
- Linear Bezier

$$r(t) = (1-t)V_0 + tV_1$$



- Quadratic Bezier

$$r(u) = (1-u)^2 V_0 + 2(1-u)uV_1 + u^2V_2$$



Bezier Curve의 일반식 (2)

$$B_{i,n}(t) = \binom{n}{i}(t)^i(1-t)^{n-i} \qquad 0 \le t \le 1$$

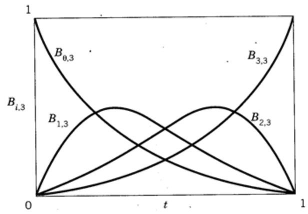
– Blending function of Cubic Bezier Curve (degree 3)

$$B_{0,3} = \frac{3!}{0!\,3!} t^0(1-t)^3 = (1-t)^3$$

$$B_{1,3} = \frac{3!}{1!\,2!} t^1(1-t)^2 = 3t(1-t)^2$$

$$B_{2,3} = \frac{3!}{2!\,1!} t^2(1-t) = 3t^2(1-t)$$

$$B_{3,3} = \frac{3!}{3!\,0!} t^3(1-t)^0 = t^3$$



■ Normalizing Property

$$[(1-t)^3] + [3t(1-t)^2] + [3t^2(1-t)] + t^3 = 1$$

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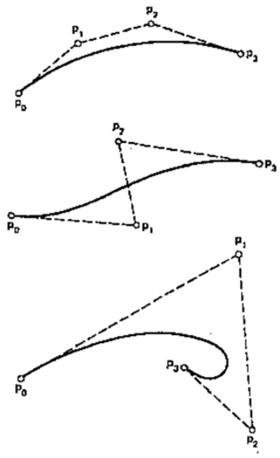
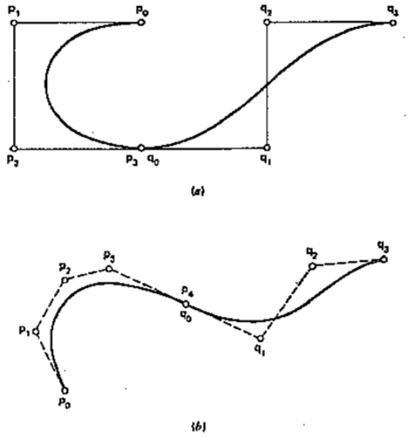
Bezier Curve의 예





Composite Bezier curves

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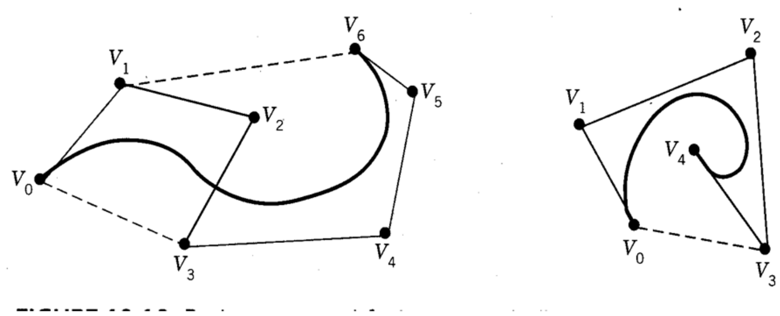
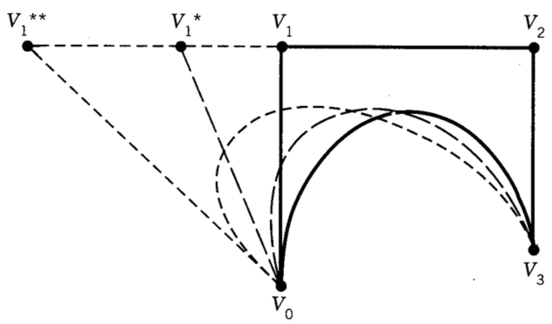
Bezier Curve의 성질 (1)

- **Convex Hull Property**



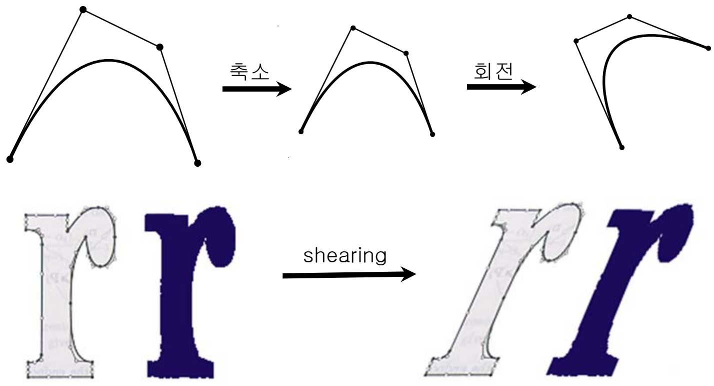
Bezier Curve의 성질 (2)

- **Effect of moving control points**



Bezier Curve의 성질 (3)

- **Affine Invariance**
  - **Control point를 transformation하면 곡선도 같이 transform된다.**



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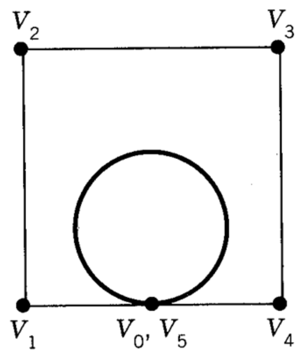
Bezier Curve의 성질 (4)

- **Closed loop : first and last control points are coincide**



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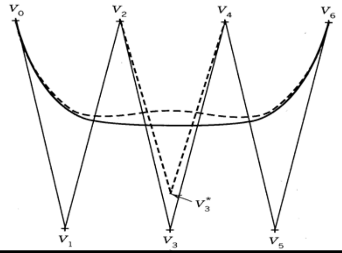
Bezier Curve의 성질 (5)

- 조정 점의 개수와 곡선의 차수가 직결되어 모든 조정 점이 곡선의 형상에 영향을 줌.

$$r(u) = \sum_{i=0}^{n} B_i^n(u) V_i, \qquad 0 \le u \le 1 \qquad B_i^n(u) = \binom{n}{i}(1-u)^{n-i} u^i$$

  - 많은 조정 점을 이용할 경우 곡선식의 차수도 올라가게 되어 계산량이 증가되며 곡선이 진동하는 문제가 야기됨
  - 모든 조정 점이 곡선의 형상에 영향을 주기 때문에 곡선의 일부분을 변형시키면 나머지 부분도 예상치 못한 변화가 발생할 수 있음.



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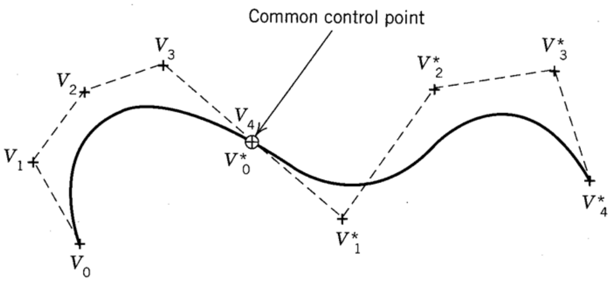
Composite Bezier Curve

- Piecewise Bezier curve in case of a large number of control points
  - C^1 continuity : three control points around the intersection are colinear



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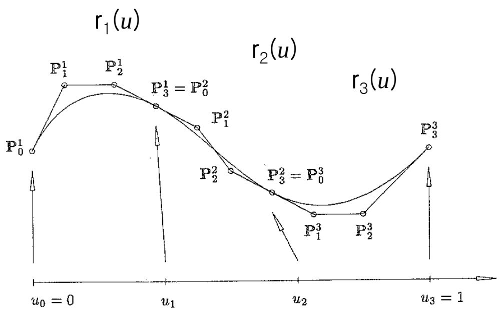
Composite Curve (복합 곡선)

- 복잡한 자유곡선을 표현하는 두가지 방법
  1) 곡선의 차수 증가 ⟶ 원하지 않는 굴곡 발생
  2) 여러개의 곡선 결합으로 표현 ⟶ 대부분의 CAD / CAM system 이용
- Composite Curve : piecewisely defined continuous curve



{ $r_1(u)$, $r_2(u)$, $r_3(u)$ } : composite curve
$r_1(u)$, $r_2(u)$, $r_3(u)$: curve segment

Continuity (연속 조건)

- Parametric Continuity

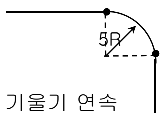
C⁰ 연속	C¹ 연속	C² 연속
$r_1(u_1) = r_2(u_1)$	$\frac{d}{du}r_1(u_1) = \frac{d}{du}r_2(u_1)$	$\frac{d^2}{du^2}r_1(u_1) = \frac{d^2}{du^2}r_2(u_1)$

- Geometric Continuity

G⁰ 연속

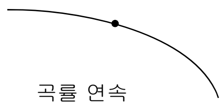
위치 연속

G¹ 연속



기울기 연속

G² 연속



곡률 연속

- G^x 연속은 curve의 재매개변수화에 의해 C^x 연속으로 바꿀 수 있다.

C^1 Continuity of Cubic Bezier Cv

- Derivative

$$Q(t) = [(1-t)^3 \quad 3t(1-t)^2 \quad 3t^2(1-t) \quad t^3]\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$Q(t) = [(1-3t+3t^2-t^3)\ (3t-6t^2+3t^3)\ (3t^2-3t^3)\ t^3]\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

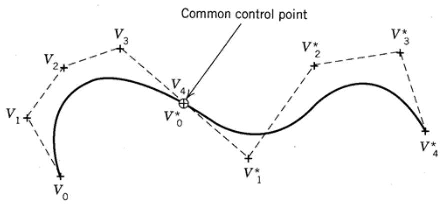
$$Q'(t) = [-3(1-t)^2 \quad 3(3t^2-4t+1) \quad -3t(3t-2) \quad 3t^2]\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

- C^1 Continuity

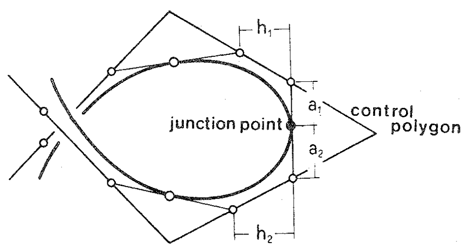
$Q'_1(1) = Q'_2(0)$

$Q'(0) = 3(V_1 - V_0)$

$Q'(1) = 3(V_3 - V_2)$



G^2 Continuity



- End point 에서의 곡률 : $k = \frac{n+1}{n}\frac{h}{a^2}$

→ G^2 연속 조건 : $\frac{a_1}{a_2} = (\frac{h_1}{h_2})^{1/2}$

B-spline Curve

1-35

B-spline Curve의 정의

$$P(u) = \sum_{i=0}^{n} P_i N_{i,k}(u) \qquad t_0 \le u \le t_{n+k}$$

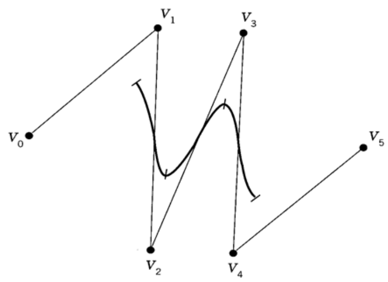
여기서,

$\{P_0, P_1, \cdots, P_n\}$; control point

$N_{i,k}(t)$: blending function of degree (k-1)

k ; B-spline의 order (degree = k−1)

t_i ; u의 범위안에 존재하는 매듭값(knot value)



0-36

B-spline Curve의 Blending Functions

$$P(u) = \sum_{i=0}^{n} P_i N_{i,k}(u) \qquad t_0 \le u \le t_{n+k}$$

여기서,

$$N_{i,k}(u) = \begin{cases} 1 & if\ t_i \le u \le t_{i+1} \\ 0 & \text{otherwise} \end{cases} \qquad t_i = \begin{cases} 0 & 0 \le i < k \\ i-k+1 & k \le i \le n \\ n-k+2 & n < i \le n+k \end{cases}$$

$$N_{i,k}(u) = \frac{(u - t_i)}{t_{i+k-1} - t_i} N_{i,k-1}(u) + \frac{(t_{i+k} - u)}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(u)$$; Cox-deBoor function

t_i ; u의 범위안에 존재하는 매듭값(knot value)

k ; B-spline의 order (degree = k-1)

A B-spline of order k in the l-th span is the weighted average of the B-splines of order (k-1) in the l-th and (l+1)st spans

- **Convex hull property**
- **Normalizing property** $\sum_{i=0}^{n} N_{i,k}(t) = 1$

0-37

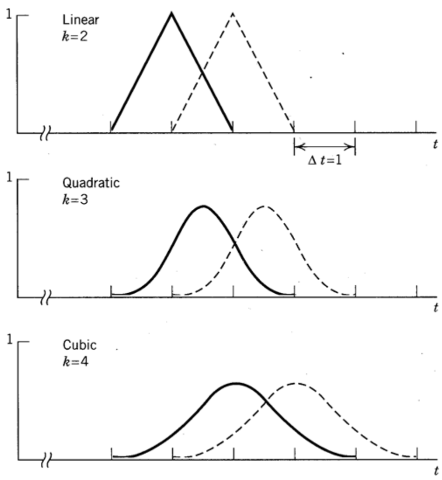
B-Spline Curve의 특성

- 조정 점의 개수와 다항식의 차수가 서로 독립적이다.
  - 설계자가 원하는 차수를 직접 정할 수 있음.
  - **Bezier curve**에서는 조정점의 개수 = 차수 +1
- 국부적인 형상 조정이 가능하다.
  - 모든 블렌딩 함수는 매개변수 **u**의 전체 범위 중 각각 서로 다른 일정 범위에서만 값을 갖도록 함.
  - **Bezier curve**에서는 블렌딩 함수가 **u**의 전체범위에서 값을 가짐
    - => 형상이 전체적으로 바뀜
- **Degree**가 **3**차 이상(**order**는 **4**차이상)이면 **2**차 미분 연속이 보장됨.

0-38

B-spline Curve의 Blending Functions

- Blending functions for uniform B-splines of various orders



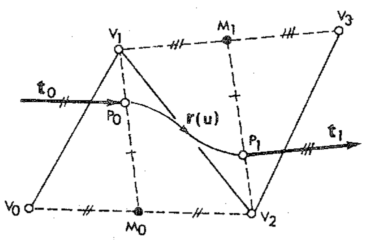
0-39

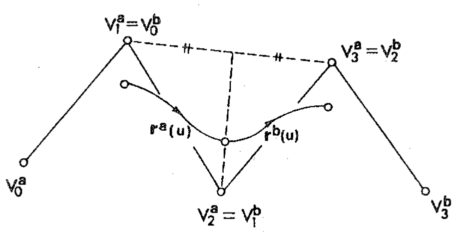
(Uniform) Cubic B-Spline Curve

- $r(u) = UNR \qquad 0 \le u \le 1$

여기서, $U = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix}$

$$N = \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -1 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} \text{ ; control points}$$





0-40

Knot Vector

- Relationship of parameters
  - a knot vector $[t_0, \ldots, t_m]$

$$\underset{\text{no. of knots}}{(m+1)} = \underset{\text{no. of control points}}{(n+1)} + \underset{\text{order of curve}}{k}$$

$$m = n + k$$

$N_{0,k}(t) \ldots N_{n,k}(t) \Rightarrow [t_0, \ldots, t_{n+k}]$

- Classification of Knot Vectors
  - Uniform / periodic
  - Nonperiodic
  - Nonuniform

0-41

Uniform/Periodic (1)

- **Uniform knot vector has equispaced t_i values**

  Let $(t_i - t_{i-1}) = a$

  [0 1 2 3 4] with a = 1

  [-0.5 0.0 0.5 1.0 1.5] with a = 0.5

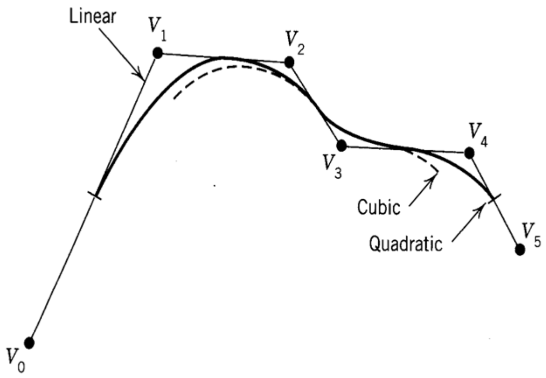
Degree $(k-1)$	Order (k)	Knot Vector $(m = n + k)$	Parameter Range $(k-1) \le t \le (n+1)$
1	2	[0 1 2 3 4 5 6 7]	$1 \le t \le 6$
2	3	[0 1 2 3 4 5 6 7 8]	$2 \le t \le 6$
3	4	[0 1 2 3 4 5 6 7 8 9]	$3 \le t \le 6$

- Normalized in the range of [0 to 1]
  - [0 1/4 1/2 3/4 1]

0-42

Uniform/Periodic (2)

- Uniform B-splines of various degrees

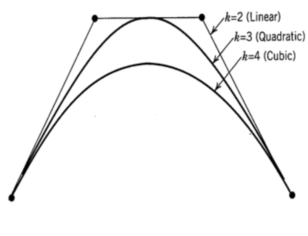


0-43

Nonperiodic (1)

- **Nonperiodic or Open Knot Vector**
  - **Has repeated knot values at the ends with multiplicity equal to the order of the function k and internal knots equally spaced**

Order (k)	No. of knots $(m = n + k)$	Nonperiodic knot vector
2	6	$[\underbrace{0\ \ 0}_{k}\ \ 1\ \ 2\ \ \underbrace{3\ \ 3}_{k}]$
3	7	$[\underbrace{0\ \ 0\ \ 0}_{k}\ \ 1\ \ \underbrace{2\ \ 2\ \ 2}_{k}]$
4	8	$[\underbrace{0\ \ 0\ \ 0\ \ 0}_{k}\ \ \underbrace{1\ \ 1\ \ 1\ \ 1}_{k}]$



- General expression

$$t_i = 0 \rightarrow i < k$$

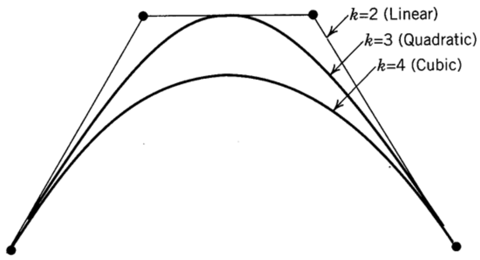
$$t_i = i - k + 1 \rightarrow k \le i \le n$$

$$t_i = n - k + 2 \rightarrow i > n$$

0-44

Nonperiodic (2)

- No loss of parameter range
  - Curve interpolates the first and last control points
  - 0 =< u =< n-k+2



0-45

Nonperiodic (3)

- The Bezier Representation
  - A special case of nonperiodic B-spline
    - If no.of control points (n+1) = order (k) and a nonperiodic uniform knot vector is used
    - Cubic B-spline with 4 control points and a knot vector [0 0 0 0 1 1 1 1]

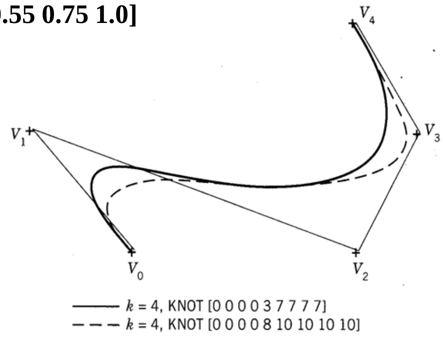
    ➔ Cubic Bezier curve

$$[\underbrace{0 \quad 0 \quad \cdots \quad 0}_{k} \quad \underbrace{1 \quad 1 \quad \cdots \quad 1}_{k}]$$

0-46

Nonuniform (1)

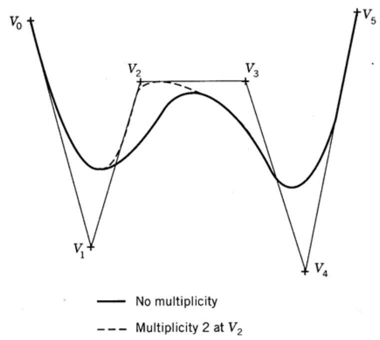
- **Multiple interior knot values or unequal spacing**
  - **[0 1 2 3 3 4]**
  - **[0.0 0.20 0.55 0.75 1.0]**



0-47

Nonuniform (2)

- Effect of multiplicity of control points
  - Generate a span of zero length
  - C_{k-m-2} continuity at ti
    - m (=< k-2) is the multiplicity of interior knot value



0-48

(Order=3) B-spline Curve의 예

P_0, P_1, P_2의 조정점을 갖고 order (k)가 3인 비주기적 B-spline 곡선

비 주기 매듭값 t_i는 다음과 같음.

$$t_0 = 0, t_1 = 0, t_2 = 0, t_3 = 1, t_4 = 1, t_5 = 1$$

K=1에 해당되는
블렌딩 함수 $N_{i,1}$을 유도.

$$N_{0,1} = \begin{cases} 1 & t_0 \le u < t_1 \\ 0 & otherwise \end{cases}$$

$$N_{1,1} = \begin{cases} 1 & t_1 \le u < t_2 \\ 0 & otherwise \end{cases}$$

$$N_{2,1} = \begin{cases} 1 & t_2 \le u < t_3 \\ 0 & otherwise \end{cases}$$

$$N_{3,1} = \begin{cases} 1 & t_3 \le u < t_4 \\ 0 & otherwise \end{cases}$$

$$N_{4,1} = \begin{cases} 1 & t_4 \le u < t_5 \\ 0 & otherwise \end{cases}$$

K=2에 해당되는
블렌딩 함수 $N_{i,1}$을 유도.

$$N_{0,2} = \frac{(u-t_0)N_{0,1}}{t_1 - t_0} + \frac{(t_2-u)N_{1,1}}{t_2-t_1} = \frac{uN_{0,1}}{0} + \frac{(-u)N_{1,1}}{0} = 0$$

$$N_{1,2} = \frac{(u-t_1)N_{1,1}}{t_2 - t_1} + \frac{(t_3-u)N_{2,1}}{t_3-t_2} = \frac{uN_{1,1}}{0} + \frac{(1-u)N_{2,1}}{1} = (1-u)$$

$$N_{2,2} = \frac{(u-t_2)N_{2,1}}{t_3 - t_2} + \frac{(t_4-u)N_{3,1}}{t_4-t_3} = \frac{uN_{2,1}}{1} + \frac{(1-u)N_{3,1}}{0} = u$$

$$N_{3,2} = \frac{(u-t_3)N_{3,1}}{t_4 - t_3} + \frac{(t_5-u)N_{4,1}}{t_5-t_4} = \frac{(u-1)N_{3,1}}{0} + \frac{(1-u)N_{4,1}}{0} = 0$$

0-49

(Order=3) B-spline Curve의 예

K=3에 해당되는 블렌딩 함수 $N_{i,1}$을 유도.

$$N_{0,3} = \frac{(u-t_0)N_{0,2}}{t_2 - t_0} + \frac{(t_3-u)N_{1,2}}{t_3-t_1} = \frac{uN_{0,2}}{0} + \frac{(1-u)N_{1,2}}{1} = (1-u)^2$$

$$N_{1,3} = \frac{(u-t_1)N_{1,2}}{t_3 - t_1} + \frac{(t_4-u)N_{2,2}}{t_4-t_2} = u(1-u) + (1-u)u = 2u(1-u)$$

$$N_{2,3} = \frac{(u-t_2)N_{2,2}}{t_4 - t_2} + \frac{(t_5-u)N_{3,2}}{t_5-t_3} = u^2$$

다음 식을 위의 값을 이용하여 정리하면 다음과 같다.

$$P(u) = \sum_{i=0}^{n} P_i N_{i,k}(u) \qquad 0 \le u \le n-k+2$$

$$P(u) = (1-u)^2 P_0 + 2u(1-u)P_1 + u^2 P_2$$

0-50

k = 4—Cubic

$$N_{i,4}(t) = \frac{(t - t_i)}{(t_{i+3} - t_i)} N_{i,3}(t) + \frac{(t_{i+4} - t)}{(t_{i+4} - t_{i+1})} N_{i+1,3}(t) \quad (10.94)$$

From Eqs. 10.91, 10.92, and 10.93 the value of $N_{i+1,3}$ can be found. Following a procedure similar to the one used for the quadratic blending functions, the cubics can be written as

$$N_{0,4}(t) = \frac{(t - t_i)^3}{(t_{i+1} - t_i)(t_{i+2} - t_i)(t_{i+3} - t_i)} \qquad t_i \le t < t_{i+1} \quad (10.95)$$

$$N_{1,4}(t) = \left.\begin{aligned} &\frac{(t - t_i)^2(t_{i+2} - t)}{(t_{i+3} - t_i)(t_{i+2} - t_i)(t_{i+2} - t_{i+1})} \\ &+ \frac{(t - t_i)(t_{i+3} - t)(t - t_{i+1})}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})} \\ &+ \frac{(t_{i+4} - t)(t - t_{i+1})^2}{(t_{i+4} - t_{i+1})(t_{i+3} - t_{i+1})(t_{i+2} - t_{i+1})} \end{aligned}\right\} \quad t_{i+1} \le t < t_{i+2} \quad (10.96)$$

$$N_{2,4}(t) = \left.\begin{aligned} &\frac{(t - t_i)(t_{i+3} - t)^2}{(t_{i+3} - t_i)(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} \\ &+ \frac{(t_{i+4} - t)(t - t_{i+1})(t_{i+3} - t)}{(t_{i+4} - t_{i+1})(t_{i+3} - t_{i+1})(t_{i+3} - t_{i+2})} \\ &+ \frac{(t_{i+4} - t)^2(t - t_{i+2})}{(t_{i+4} - t_{i+1})(t_{i+4} - t_{i+2})(t_{i+3} - t_{i+2})} \end{aligned}\right\} \quad t_{i+2} \le t < t_{i+3} \quad (10.97)$$

$$N_{3,4}(t) = \frac{(t_{i+4} - t)^3}{(t_{i+4} - t_{i+1})(t_{i+4} - t_{i+2})(t_{i+4} - t_{i+3})} \qquad t_{i+3} \le t \le t_{i+4} \quad (10.98)$$

B-Spline Formulation (Highlights)

B-Spline General Form

$$P(t) = \sum_{i=0}^{n} N_{i,k}(t) V_i$$

defining a point on the curve

where i indicates position: $V_i \rightarrow$ control point i

$t_i \rightarrow$ knot i

k is order of curve

$(n + 1)$ is number of control points

$N_{i,k}(t)$ are blending functions given by Eq. 10.82

V_i are control points

Continuity

Continuity of position, C^0

[1 to $(k - 2)$] derivatives are continuous

0-52

Knot Vectors

These are the parametric intervals within which the blending functions are defined. For example:

$$[t_0 \,.\,.\,.\,.\,.\, t_m]$$

The elements "t_i" of the knot vector must satisfy the relation:

$$t_i \leq t_{i+1}$$

The relationship among

degree of the curve $(k - 1)$
no. of control points $(n + 1)$
no. of knots $(m + 1)$

is given by

$$m = n + k$$

0-53

Classification of Knot Vectors:

I. Uniform/Periodic

$(t_i - t_{i-1}) =$ constant
Influence of each basis function is limited to k (see Figure 10.34).
Parameter range: $(k - 1) \leq t \leq (n + 1)$

Example: [0 1 2 3 4]

II. Nonperiodic

Satisfies the following equations:

$t_i = 0$	for	$i < k$
$t_i = i - k + 1$	for	$k \leq i \leq n$
$t_i = n - k + 2$	for	$i > n$

for a knot vector starting at $i = 0$ and $(n + 1)$ control points.
No loss of parameter range, so that the curve interpolates the first and last control points.

Example: [0 0 1 2 2 3 3

III. Nonuniform

Knot vector is not equispaced.

Example: [0 1 2 3 3 4]

0-54

Bezier Representation

It is simply a special case of the general B-spline formulation when the following conditions are satisfied:

- The number of defining polygon vertices is equal to the order of the B-spline basis.
- A nonperiodic knot vector is used.

For example:

No. of control points = 4 (i.e., $n = 3$)

Order of the curve, $k = 4$

Open periodic knot vector: [0 0 0 0 1 1 1 1]

0-55

Rational Curves

- **General Meaning**
  - **Functions are obtained by the "ratio" of two polynomials**
  - **This representation make use of the concept of homogeneous coordinates**
  - 한 꼭지점이 곡선에 미치는 영향의 양을 결정할 수 있음
- **General Form**

	Bezier	B-Spline
Nonrational (Integral)	$Q(t) = \sum_{i=0}^{n} B_{in}(t) V_i$	$P(t) = \sum_{i=0}^{n} N_{ik}(t) V_i$
Rational	$Q(t) = \dfrac{\sum_{i=0}^{n} B_{in}(t) w_i V_i}{\sum_{i=0}^{n} B_{in}(t) w_i}$	$P(t) = \dfrac{\sum_{i=0}^{n} N_{ik}(t) w_i V_i}{\sum_{i=0}^{n} N_{ik}(t) w_i}$

0-56

Conic vs Rational Quadratic Polynomial Curve

- n차 polynomial curve

  : 차수를 아무리 높여도 conic curve를 근사적으로 밖에 표현하지 못함

- Rational polynomial curve

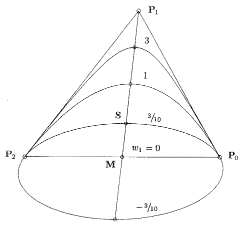
  : 2차(quadratic)로 모든 종류의 conic curve를 정확히 표현함

⇨ NURB(Non-Uniform Rational B-spline)가 널리 쓰이는 이유임.

0-57

Quadratic Rational Polynomial Curves

- Conic arcs



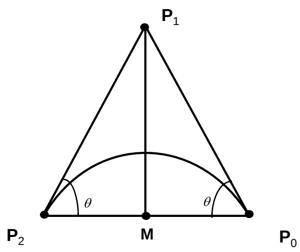
$w_0 = w_2 = 1,$

$w_1 = 1$: parabola

$0 \le w_1 \le 1$: ellipse

$w_1 > 1$: hyperbola

- Circular arc



$w_0 = w_2 = 1$

$w_1 = \cos\theta$

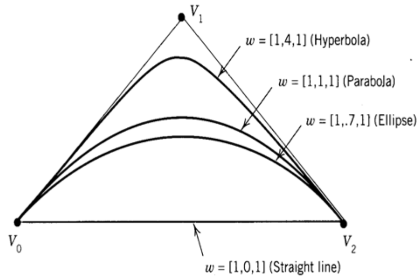
0-58

Quadratic Rational B-Spline

k = 3 => (order)
n = 2 => (no. of control vertices = n+1 = 3)
m = 5 => (n+k)
knot vector [t] = [0 0 0 1 1 1]
weight [w] = [1 w_1 1]

$$P(t) = \frac{N_{0,3}(t)w_0V_0 + N_{1,3}(t)w_1V_1 + N_{2,3}(t)w_2V_2}{N_{0,3}(t)w_0 + N_{1,3}(t)w_1 + N_{2,3}(t)w_2}$$

$$P(t) = \frac{N_{0,3}(t)V_0 + N_{1,3}(t)w_1V_1 + N_{2,3}(t)}{N_{0,3}(t) + N_{1,3}(t)w_1 + N_{2,3}(t)}$$



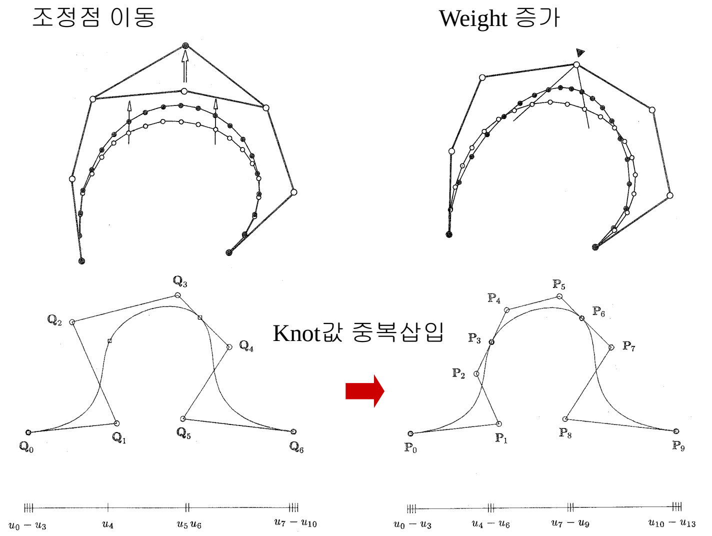
$w_1 = 0$ straight line
$0 < w_1 < 1$ elliptic segment
$w_1 = 1$ parabolic segment
$w_1 > 1$ hyperbolic segment

$w_1 = \cos\theta$: cir cle

NURBS Curve

- Non-Uniform Rational B-spline Curve
- 가장 일반적 형태의 B-spline curve
- NURB curve data (in IGES)
  - p : degree
  - n : highest index of control points (= *number* − 1)
  - P_0, P_1, … , P_n : Euclidean control points
  - $w_0, w_1, \ldots, w_n$: weights
  - $u_0, u_1, \ldots, u_m$: knot vector ($m = n + p + 1$)
  - s_0, s_1 : start and end parameter values ($u_0 \le s_0 < s_1 \le n_m$)
  - 참고사항
    - planar or nonplanar
    - open or closed
    - rational or nonrational
    - nonperiodic(clamped) or periodic(unclamped)

Knot, Control Point, Weight의 영향



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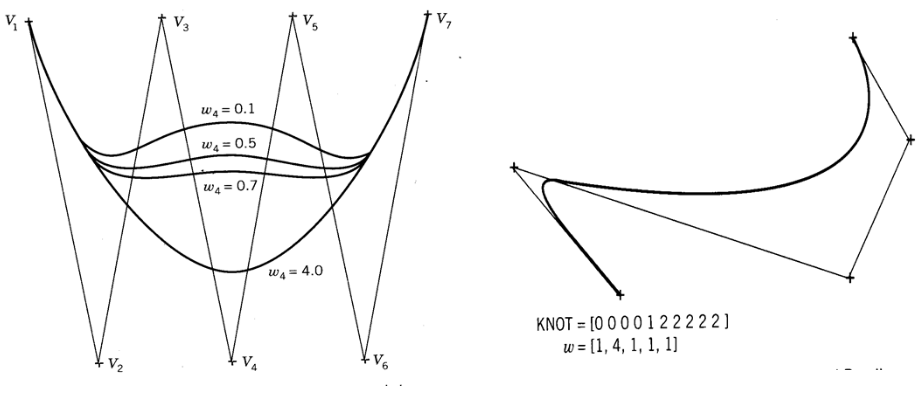
Effect of Weight

- Nonperiodic cubic rational B-spline



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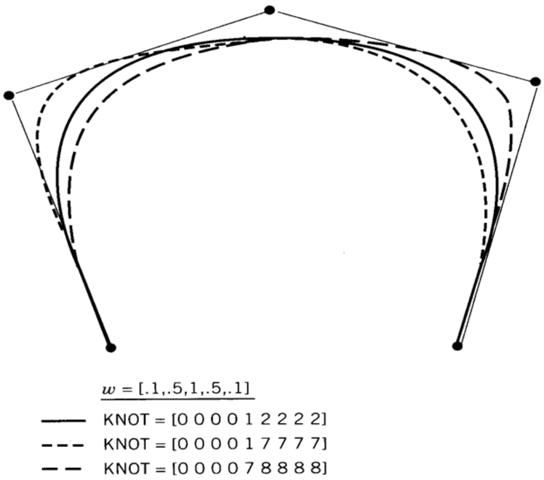
Effect of Knot Vector

- Examples of Cubic NURBS



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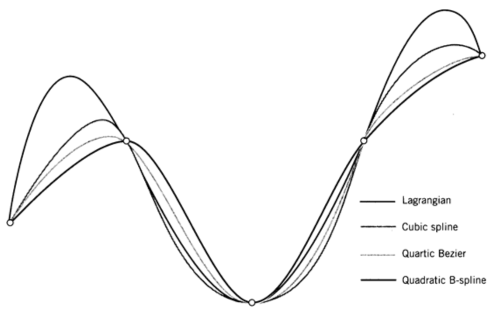
Comparison of Various Curve-Defining Techniques

- Lagrange polynomials
- Natural cubic spline
- Quartic Bezier curve
- Nonperiodic, quadratic B-spline curve

Input points: (0,0) (1,1) (2,-1) (3,1) (4,2)



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